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ON ALFA-PARTICLE HEATING OF A THERMONUCLEAR PLASMA

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ABSTRACT

The theory of the slowing down and thermalization of alfa-particles created in and slowed down by a plasma considered as an undisturbed thermal bath is re-examined. The considerations are limited to a homogeneous, isotropic plasma. Also the alfa-particles are considered as a density-wise negligible component of the plasma. Under these circumstances, all essential parts of the theory can be studied analytically. Thus all standard approximations can be studied with precision. However, the conclusions do not differ from numerical results earlier obtained by numerous authors. In this sense the present paper must be considered as a review based on analytical methods.

The standard approximations referred to are such as the neglect of the variation of the Coulomb logarithm with velocity, the influence of quantum-mechanical effects, the significance of the impact parameter cut-off (the Debye length), and the assumption of a Maxwellian plasma. Some of these effects have been examined analytically by other authors. However, the present work shows that the Fokker-Planck equation can be solved without the aid of computers for the simplest set of standard approximations. The result thereof is a determination of slowing down rates and rate of growth of energy spread and thereby thermalization of the alfa-particles.

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CONTENTS

INTRODUCTION

The theory of the slowing down of energetic particles in a plasma has been re-examined in view of the ultimate goal of the JET experiment[1), i.e. the investigation of the slowing down of alfa-particles resulting from deuterium tritium fusion in the JET plasma. Such an experimental investigation is planned as one of the last exercises to be conducted on JET.

The slowing down of energetic particles in a plasma is described in the classical paper by Rosenbluth, MacDonald and Judd[2)]. Several authors have repeated and expanded such studies. However, the present paper will quote but a few[3)].

Computations of friction and diffusion coefficients from the theory of Coulomb collisions are carried further than usual in the sense that the various causes of the possible variation of the Coulomb logarithm, log Λ, are examined in detail. The results are, however, as expected. For parameters of interest to a thermonuclear plasma, neither variations of temperature, density, Debye-length cut-off, nor of the precise point of transition to quantum-mechanical considerations are of great significance for the friction and diffusion coefficients (except for the obvious scaling laws resulting in the log Λ = constant approximation).

In this same approximation (log Λ = constant) and neglecting the so-called small terms in the diffusion coefficients, and for a homogeneous, isotropic and thermalized plasma, the Fokker-Planck equation may be expressed in a particularly simple form permitting considerable insight using analytical methods and avoiding numerical work. A certain number of simple explanations thus become evident.

THE INITIAL SITUATION

The initial situation consists of the creation of an alfa-particle through the reaction

$$D + T \rightarrow \alpha + n \tag{1}$$

We assume this to happen in a homogeneous, isotropic and electrically neutral plasma consisting of equal densities of deuterium and tritium, i.e.

$$n_D = n_T = n_e/2 \tag{2}$$

with n_D, n_T, and n_e being the deuterium, tritium, and electron densities, respectively.

The alfa-particle created in reaction (1) is born with a kinetic energy of E_α = 3.52 MeV in the rest frame of the reacting deuterium-tritium system. We could, of course, easily transform to the laboratory system and obtain an initial distribution of alfa-particle energies including the Doppler-broadening resulting from the thermal motion of deuterium and tritium. However, we shall simplify and assume an initial situation describable in terms of a delta function in energy at 3.52 MeV.

We shall, as mentioned, assume that the alfa-particles have a negligible influence on the plasma at large. Thus the velocity distributions of the electrons and the tritium and deuterium components are constant in time. We use the notation $P_i(v_i)$ for the i^{th} component and normalize to unit density

$$\int_0^\infty P_i(v_i)\, v_i^2 dv_i = 1 \tag{3}$$

The velocity distribution $F(t,v)$ of our alfa-particle depends on time, t, and it is precisely this development in time that we wish to study. At time t = 0 we have, as mentioned above,

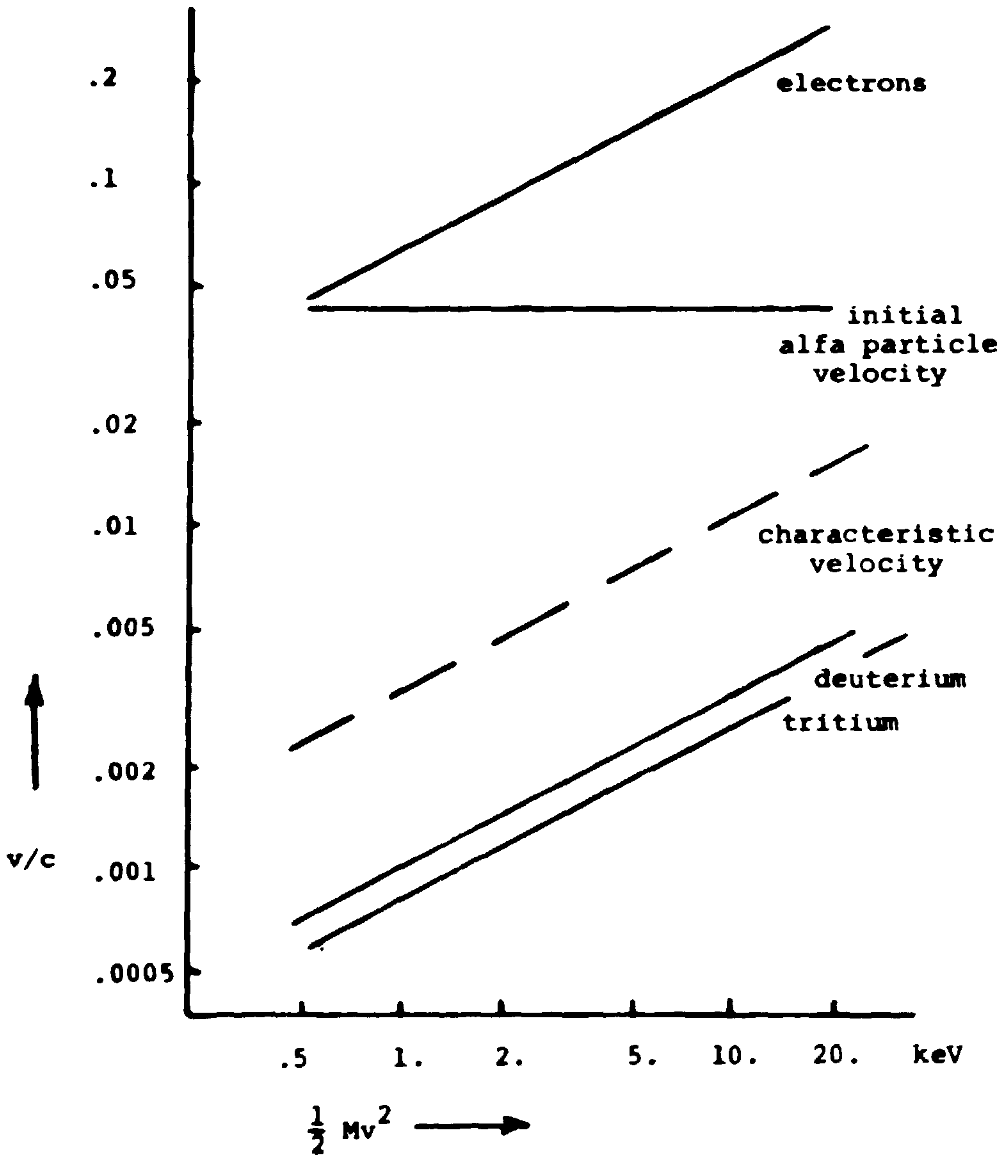
$$F(o,v) = \frac{1}{v_o^2}\,\delta(v-v_o) \tag{4}$$

where v_o is the alfa-particle velocity corresponding to the kinetic energy of 3.52 MeV, as mentioned.

It is of considerable consequence that the initial velocity of our α-particle is typically slightly less than the average electron velocity but very much greater than the average ion velocity for the plasmas of interest (average particle energy of a few keV). This is due to the smallness of the mass ratios M_e/M_D etc. The point in question is illustrated in Fig. 1, which shows velocity to energy relations for e, D and T and where the initial α-particle velocity is also indicated.

As a consequence the α-particle is initially cooled by the electrons only. As can be estimated following, e.g., the discussion of Sivukhin[4)], the energy loss to electrons and ions breaks approximately even for $E_\alpha \cong 20\ E_e$ for a 50%-50% DT mixture. This characteristic velocity is also marked on the figure*. From this it is evident that high plasma temperature shifts the energy dissipation so as to deliver a larger fraction into ion heating. However, it also slows down the energy loss in time.

*See also eq. (56).



<u>Fig. 1</u>. The figure shows velocity divided by velocity of light, c, vs. kinetic energy $\frac{1}{2}$ Mv^2. Also shown is the initial velocity of the α-particles from DT fusion.

THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation appropriate to our slowing down problem can be written as follows (in Cartesian coordinates in velocity space)

$$\frac{\partial F}{\partial t} = \sum_{\beta} \left[- \nabla(F \cdot \bar{\Delta}_{\beta}) + \frac{1}{2} \nabla_i \nabla_k (F \cdot D_{ik}^{(\beta)}) \right] \tag{5}$$

with the friction on the component β (for D, T and e) given by

$$\bar{\Delta}_{\beta} = \Delta^{(\beta)}(v)\hat{v} \tag{6}$$

and the diffusion on species β given by

$$D_{ik}^{(\beta)} = D_1^{(\beta)} \underline{\underline{1}} + D_2^{(\beta)} \hat{v}_i \hat{v}_k \tag{7}$$

with the notation $\hat{v}$ for unit vectors, and $\hat{v}_i \hat{v}_k$ being the dyadic formed by two unit vectors. The form of eqs. (6) and (7) is a consequence of homegeneity and isotropy. Only one direction is specified in space, i.e. that of the α-particle velocity $\bar{v}$. Thus any vector coefficient must point in the direction $\hat{v}$ and any tensor must be describable as an ellipsoid with one component parallel to $\hat{v}$ and one perpendicular to $\hat{v}$, the latter with azimuthal symmetry around $\hat{v}$. Such a tensor can be written in the form (7). We now note that F, Δ, D_1 and D_2 depend on the numerical value of v only. Carrying out the differentiations involved in (5) through v (or using formulas from curvilinear geometry)[5] we obtain

$$\frac{\partial F}{\partial t} = \sum_{\alpha} \left[- \frac{1}{v^2} \frac{\partial}{\partial v} (v^2 F \Delta^{(\beta)}) + \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial v} v^2 \frac{\partial}{\partial v} (F D_1^{(\beta)}) + \frac{1}{2} \frac{1}{v^2} \frac{\partial^2}{\partial v^2} (v^2 F D_2^{(\beta)}) \right] \tag{8}$$

where the coefficients must now be specified.

THE FRICTION AND DIFFUSION COEFFICIENTS

We calculate the friction and diffusion coefficients from binary collision theory for Coulomb encounters. A test particle (our α-particle) of velocity $\bar{v}_1$, mass M_1 and charge Z_1 collides with a field particle (any one of our plasma particles) of velocity $\bar{v}_2$, mass M_2 and charge Z_2. The differential cross-section is given by

$$d\sigma(\theta) = \frac{Z_1^2 Z_2^2 e^4 2\pi d\cos\theta}{\mu^2 w^4 (1-\cos\theta)^2} \tag{9}$$

where θ is the scattering angle in the centre of mass (CM) system, μ is the reduced mass

$$\mu = \frac{M_1 M_2}{M_1+M_2} \tag{10}$$

and w is the relative velocity

$$w^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\phi \tag{11}$$

with ϕ the angle in the laboratory system between $\bar{v}_1$ and $\bar{v}_2$. Thus we must compute the transition-probability-averaged velocity increments per unit time

$$\Delta^{(2)}(v_1) = \int (\Delta v_{1\,||})^{(2)}\, P(v_2) v_2^2 dv_2 \cdot \tfrac{1}{2}\, d\cos\phi\; n_2 w d\sigma \tag{12}$$

$$D_1+D_2 = \int \left[(\Delta v_{1||})^{(2)}\right]^2 P(v_2) v_2^2 dv_2 \cdot \tfrac{1}{2}\, d\cos\phi\; n_2 w d\sigma \tag{13}$$

$$D_1+3D_2 = \int \left[(\Delta v_1)^{(2)}\right]^2 P(v_2) v_2^2 dv_2 \cdot \tfrac{1}{2}\, d\cos\phi\; n_2 w d\sigma \tag{14}$$

with the velocity increments obtainable from the laws of the conservation of energy and momentum.

CONSERVATION OF ENERGY AND MOMENTUM

The variables introduced so far (v_2, ϕ and θ) are carefully chosen so that their physical regicrs

$$\theta_{min} \leq \theta \leq \pi \tag{15}$$

$$0 \leq \phi \leq \pi \tag{16}$$

$$0 \leq v_2 \leq \infty \tag{17}$$

are independent of all azimuthal variables. This is possible because of isotropy and homogeneity. Figure 2 illustrates the Cartesian coordinate system with $\hat{v}_1$ as the first axis, the second axis in the plane containing $\bar{v}_1$ and $\bar{v}_2$ so that the third axis, $\hat{e}_3$, lies in the direction of the vector product $\bar{v}_1 \times \bar{v}_2$. We introduce a unit vector $\hat{\alpha}$ in the 1,2 plane and perpendicular to the relative velocity $\bar{w} = \bar{v}_1 - \bar{v}_2$, and we note that the impact parameter $\bar{\rho}$ lies in the $\hat{\alpha}$, $\hat{e}_3$ plane. Let $\bar{r}_{12}$ be the relative distance of the two particles before the encounter, and let ψ be the azimuthal angle of $\bar{r}_{12}$ projected onto the $\hat{\alpha}$, $\hat{e}_3$ plane. It then follows that

$$\hat{\rho} = \hat{\alpha}\cos\psi + \hat{e}_3\sin\psi \tag{18}$$

Because of homogeneity, ψ is a trivial variable with the independent physical region

$$0 \leq \psi \leq 2\pi \tag{19}$$

and with the constant probability distribution $\frac{1}{2\pi}$ per unit interval $d\psi$ (transition-probability 1 per unit $d\psi$).

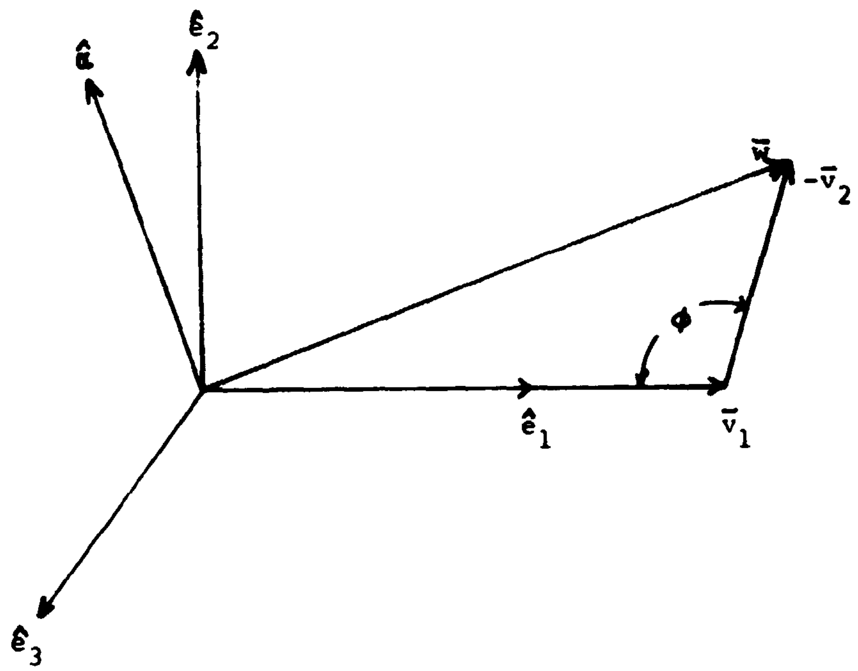


Fig. 2. A convenient coordinate system. The unit vector $\hat{e}_3$ goes out of the plane of the paper. All other vectors shown lie in the plane of the paper.

Conservation of energy and momentum tells us that elastic collisions lead to a change of the direction of $\bar{w}$ by an angle (which we call θ) in the plane containing $\hat{\rho}$ and $\hat{w}$, and when the process is considered in the CM system. Thus we transform $\bar{v}_1$ from the coordinate system of Figure 1 to the CM system, rotate by the angle θ in the $\hat{w}$, $\hat{\rho}$ plane and transform back again to the original system. We subtract and have

$$\overline{\Delta v}_1 = - \frac{M_2}{M_1+M_2} (\bar{w}(1-\cos\theta) - w\hat{\rho}\sin\theta) \tag{20}$$

Because of homogeneity and isotropy, our interest in this expression is limited to finding transition-probability averages of $(\overline{\Delta v}_1)_{||}$, $(\overline{\Delta v}_1)^2$ and $((\overline{\Delta v}_1)_{||})^2$, yielding simply

$$(\overline{\Delta v}_1)_{||} = \frac{M_2}{M_1+M_2}\left[(v_1-v_2\cos\phi)(\cos\theta-1) + v_2\sin\phi\cos\psi\sin\theta\right] \quad (21)$$

leading to averages over ψ as follows

$$<(\overline{\Delta v}_1)_{||}>_\psi = \frac{M_2}{M_1+M_2}(v_1-v_2\cos\phi)(\cos\theta-1) \quad (22)$$

and

$$<(\overline{\Delta v}_1)^2_{||}>_\psi = \left(\frac{M_2}{M_1+M_2}\right)^2\left[\left(w^2 - \frac{3}{2}v_2^2\sin^2\phi\right)(1-\cos\theta)^2 + v_2^2\sin^2\phi(1-\cos\theta)\right] \quad (23)$$

together with

$$<(\overline{\Delta v}_1)^2>_\psi = \left(\frac{M_2}{M_1+M_2}\right)^2 2w^2(1-\cos\theta) \quad (24)$$

appropriate for insertion into eqs. (12-14).

Before we carry out the integrations, we return to the physical region of θ as given in eq. (15). Here we introduced the cut off θ_{min} in order to avoid the divergency in the integrals over $d\sigma$ as given in eq. (9).

THE COULOMB LOGARITHM

In the following we shall encounter two types of integrals over θ, namely

$$I_1 = \int_{-1}^{1-\varepsilon}\frac{d\cos\theta}{1-\cos\theta} = \log\left(\frac{2}{1-\cos\theta_{min}}\right) = \log\frac{2}{\varepsilon} \equiv \log\Lambda \quad (25)$$

and

$$I_2 = \int_{-1}^{1-\varepsilon} d\cos\theta = 2-\varepsilon \quad (26)$$

Because of the rather large values of log $(2/\varepsilon)$ that will be encountered ($\sim$ 50), we call terms containing I_1 the large terms and those containg I_2 the small terms. In eq. (26) we shall simply neglect ε because that is only a correction to small terms. The result of eq. (25) is the Coulomb-logarithm. This integral contains a considerable amount of difficult physics connected with the fact that we talk only of binary collisions and neglect collective motion in the plasma. Furthermore, we look at the test particle along an average (eikonal) orbit and neglect its meanderings. For these reasons it is customary to consider the cut off where the impact parameter equals the Debye length, i.e., the length where the effect of a single charge is shielded by collective adjustment to average neutrality. The Debye length is given by

$$\lambda_D \simeq \sqrt{kT/(8\pi n e^2)} \tag{27}$$

with T the plasma temperature and k the Boltzmann constant. We then have (from the scattering law)

$$\mathrm{tg}\,\frac{\theta_{min}}{2} = \frac{Z_1 Z_2 e^2/(\frac{1}{2}\,\mu w^2)}{2\lambda_D} \tag{28}$$

and

$$\log \Lambda = \log\left(1 + \left[\frac{2\lambda_D}{Z_1 Z_2 e^2/(\frac{1}{2}\,\mu w^2)}\right]^2\right) \tag{29}$$

dependent on w.

For large velocities, w, quantum-mechanical effects set in. They can be taken into account by replacing eq. (29) by

$$\log \Lambda = \log\left(1 + \left[\frac{2\lambda_D}{\hbar/(\mu w)}\right]^2\right) \tag{30}$$

where $\hbar$ is Planck's constant divided by 2π. Equation (30) should be used when

$$\frac{\hbar}{\mu w} > \frac{Z_1 Z_2 e^2}{\frac{1}{2}\,\mu w^2} \tag{31}$$

It is now evident from the fact that the values of log Λ that we shall encounter are large, and from the fact that $\frac{1}{2}\,\mu w^2$ has characteristic values, say, of order kT*, that even relatively large variations in w change log Λ by relatively small amounts. For this reason, it is customary to take log Λ outside the integral and calculate it as an approximate average. We shall study the results of this in detail in appendix B. However, our next step will be to let a rough average of log Λ enter into the characteristic scaling laws for our problem.

SCALING THE FOKKER-PLANCK EQUATION

We now introduce a standard case to which variations of the computational approximations shall be referred. We assume that there is one kind of field particles only, and that their velocities have a thermal distribution $P(v_2)$, and we extract the Coulomb logarithm from the integrals. At the same time we transform all velocities according to

$$u = \sqrt{\frac{M_2}{2kT_2}}\; v \tag{32}$$

and write $(v_1, v_2, w) \rightarrow (u_1, u_2, \omega)$. We also transform time t as

$$\tau = Z_1^2 Z_2^2 e^4 n_2 \left(\frac{\pi}{2\sqrt{2}} \ln \Lambda\right) \cdot (M_2/(kT_2))^{3/2}/M_1^2 \cdot t \tag{33}$$

Introducing all this into the Fokker-Planck equation (8) and into the definitions of the friction and diffusion coefficients through eqs. (9), (12-14) and (22-24), we find

---

*Still better of the order of $(\frac{3}{2}\,kT + \frac{1}{2}\,\mu v_1^2)$, but we deliberately neglect this v_1 dependence.

$$\frac{\partial F}{\partial \tau} = \frac{1}{u^2}\frac{\partial}{\partial u}\left[2\,\frac{M_1+M_2}{M_2}\,u^2 J_2 F + u^2\,\frac{\partial}{\partial u}\,(FG) - \frac{\partial}{\partial u}\,(u^2 FH)\right] \tag{34}$$

with J_2, G and H being the functions that scale, and which are given by

$$J_2 = \int \frac{4}{\sqrt{\pi}}\, e^{-u_2^2}\, u_2^2 du_2\, \frac{1}{2}\, d\cos\phi\ \ (u_1 - u_2\cos\phi)/\omega^3 \tag{35}$$

$$G = (J_1 - J_3)/2 + \text{small term} \tag{36}$$

$$H = (J_1 - 3J_3)/2 + \text{small term} \tag{37}$$

$$J_1 = \int \frac{4}{\sqrt{\pi}}\, e^{-u_2^2}\, u_2^2 du_2\, \frac{1}{2}\, \cos\phi\ \ 2\omega^2/\omega^3 \tag{38}$$

$$J_3 = \int \frac{4}{\sqrt{\pi}}\, e^{-u_2^2}\, u_2^2 du_2\, \frac{1}{2}\, d\cos\phi\ \ u_2^2 \sin^2\phi/\omega^3 \tag{39}$$

Neglecting small terms, we find a few useful relations* (see appendix A)

$$G - H = J_3 \tag{40}$$

$$J_2 = u_1 J_3 \tag{41}$$

and

$$\frac{\partial}{\partial u} J_3 - \frac{2H}{u} = -2uJ_3 \tag{42}$$

which permit us to write eq. (34) in the following form (for a one-component plasma)

$$\frac{\partial F}{\partial \tau} = \frac{1}{u^2}\frac{\partial}{\partial u}\left[u^2 J_3 \left[\frac{\partial}{\partial u}\, F + 2u\,\frac{M_1}{M_2}\, F\right]\right] . \tag{43}$$

---

*Eq. (42) can be generalized to an arbitrary velocity distribution for the field particles, see eq. (76).

This equation shows that the equilibrium distribution of the α-particle velocity when slowed down in a thermal "bath" is ∿ $e^{-u^2 M_1/M_2}$, i.e. a Maxwellian with the temperature of the "bath", as is seen by reintroducing v through eq. (32).

Thus we have found that our approximations (log Λ = const, neglect of small terms, and eikonal treatment of friction and diffusion as binary collision effects) still permit exact thermalization to be contained in the formalism.

SOLUTIONS TO THE FOKKER-PLANCK EQUATION

In the following we shall solve eq. (43) with its initial value problem as described above. We shall use the momentum method. We find quite generally

$$\frac{\partial}{\partial\tau}\int_0^\infty u^{2n+2}F du = \int_0^\infty u^{2n}\frac{\partial}{\partial u}\left(u^2 J_3\left[\frac{\partial}{\partial u}F + 2u\frac{M_1}{M_2}F\right]\right)du \quad (44)$$

For n = 0 this simply means constancy of the normalization integral since the right-handed side can be integrated directly and yields zero because F vanishes rapidly (as e^{-u^2} or faster) for large n, and because both F and J_3 are well-behaved for u → 0. We normalize to unity and have

$$\int_0^\infty u^2 F du = 1 \quad (45)$$

Next we integrate in eq. (44) by parts (twice) and introduce the normalization (45). We may then write for the $2n^{th}$ moment

$$\frac{\partial}{\partial\tau}\langle u^{2n}\rangle \equiv \frac{\partial}{\partial\tau}\int_0^\infty u^{2n+2}F du$$

$$= -4n\frac{M_1}{M_2}\int_0^\infty u^{2n+2}J_3 F du + 2n\int_0^\infty F[(2n+1)u^{2n}J_3 + u^{2n+1}\frac{\partial}{\partial u}J_s]du \quad (46)$$

This equation can now be solved in the cases of interest 1) the case where $F(u)$ is significantly different from zero for $u < 1$ only, and 2) the case when $F(u)$ is significantly different from zero near u_o only and where $u_o >> 1$. We begin with the first case which corresponds to electron-cooling of the α-particles. The second case corresponds to cooling by the ions.

ELECTRON COOLING

When we are interested in small values of u only, then we may use the power series expansion of J_3

$$J_3 = \frac{4}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{2m+3} u^{2m} \tag{47}$$

When this is inserted into eq. (46), we obtain an infinite set of linear first order differential equations relating all the moments of u^2. We note that higher moments contain faster time constants through the factor 4n contained in the right hand side of eq. (46). Thus the longest time effects are included in even the most drastic truncation of the set of equations*. We thus use the crude approximation only leading to

$$\frac{\partial}{\partial \tau} \langle u^2 \rangle = \frac{16}{3\sqrt{\pi}} \frac{M_1}{M_2} \left[\frac{3}{2} \frac{M_2}{M_1} - \langle u^2 \rangle + .. \right] \tag{48}$$

$$\frac{\partial}{\partial \tau} \langle u^4 \rangle = \frac{32}{3\sqrt{\pi}} \frac{M_1}{M_2} \left[\frac{5}{2} \frac{M_2}{M_1} \langle u^2 \rangle - \langle u^4 \rangle + .. \right] \tag{49}$$

which are easily solved. Assume the initial situation to be $u = u_o$, i.e. a δ-function distribution. Then

---

*This statement is justified in detail in Appendix C.

$$\langle u^2\rangle = \frac{3}{2}\frac{M_2}{M_1} + (u_o^2 - \frac{3}{2}\frac{M_2}{M_1})\ e^{-\tau 16M_1/(3\sqrt{\pi}M_2} \tag{50}$$

$$\langle u^4\rangle - \langle u^2\rangle^2$$

$$= 2\frac{M_2}{M_1}\ (1-e^{-\tau 16M_1/(3\sqrt{\pi}M_2}) \left[(u_o^2-\frac{3}{4}\frac{M_2}{M_1})e^{-\tau 16M_1/(3\sqrt{\pi}M_2)} + \frac{3}{4}\frac{M_2}{M_1} \right] \tag{51}$$

Note that the truncation is a truncation in both powers of u_o and of M_2/M_1. Note that the truncation has been done in such a fashion that the final equilibrium situation obtains, since we know that this is correct to all orders. The results are consequently correct both at $\tau = 0$ and for $\tau \to \infty$. We have neglected small term phenomena of short "decay" times.

Physically, we find the "decay" time

$$\bar{t}_e = \frac{3}{4}\sqrt{\frac{2}{3}}\ \frac{(kT_2)^{3/2}M_1}{\sqrt{M_2}\ \log\Lambda\ Z_1^2 Z_2^2 e^4 n_2} \tag{52}$$

which is unfortunately a long time, because of the factor $\sqrt{M_2}$ in the denominator. We also see that the energy spread, eq. (51), is small relative to the energy eq. (52), again because of the small electron mass. We now turn to the second case.

ENERGY DEPOSITED IN IONS

The energy loss to the ions is initially very small since $u_o \gg 1$. In this case we may insert the asymptotic expansion for J_3 into equation (46). The result is easily evaluated for the case where the electron cooling can be considered as resulting in a δ-function in u^2 centered around $\langle u^2\rangle$ given by eq. (50), and with the ion-cooling being a small perturbation on this result. We insert (for n = 1)

$$J_3 \simeq 1/u^3 \tag{53}$$

into eq. (46) and think of u^2F as a δ-function around u_o. Note that the two last terms cancel each other exactly. The result is

$$\frac{\partial}{\partial\tau} \log \langle u^2\rangle_o = -4 \frac{M_1}{M_2} \frac{1}{u_o^3} \tag{54}$$

with the time constant $\bar{t}_{ion}$, which we compare to $\bar{t}_e$ given by eq. (52). We find

$$\left.\frac{\bar{t}_e}{\bar{t}_{ion}}\right|_{mean} = \frac{3\sqrt{\pi}}{4} \frac{M_e}{M_{ion}} \frac{\log \Lambda_{ion}}{\log \Lambda_e} \left(\frac{M_\alpha}{M_e} \frac{kT_e}{E_\alpha}\right)^{3/2} \tag{55}$$

where we have used $n_e = n_D + n_T$, and where M_{ion} should be taken as $(M_D+M_T)/2$. The initial situation where the content of the bracket is close to 1 is clearly that where almost all energy goes to the electrons. At the so-called critical energy

$$E_{crit} = \frac{M_\alpha}{M_e} kT_e \left[\frac{3\sqrt{\pi}}{4} \frac{M_e}{M_{ion}} \frac{\log \Lambda_{ion}}{\log \Lambda_e}\right]^{2/3} \tag{56}$$

the ions become dominating as regards energy loss mechanism.

It is useful to write down the equations for $\langle u^2\rangle$ and $\langle u^4\rangle$ when J_3 is described by eq. (53). From eq. (46) we find

$$\frac{\partial}{\partial\tau} \langle u^2\rangle = -4 \frac{M_1}{M_2} \langle\frac{1}{u}\rangle \tag{57}$$

$$\frac{\partial}{\partial\tau} \langle u^4\rangle = -8 \frac{M_1}{M_2} \langle u\rangle + 8\langle\frac{1}{u}\rangle \tag{58}$$

These equations are, of course, only valid when $u > 1$, i.e. some time before thermalization.

TWO-COMPONENT PLASMA

Now we take a look at the energy spread when both electrons and ions are taken into account. We base the effect of the electrons on the approximations given in eqs. (48) and (49) with $M_2 = M_e$, but now we wish to undo the scaling with masses (for simplicity, we assume $T_e = T_{ion}$ and $\Lambda_e = \Lambda_{ion}$). Thus we introduce (cf. eqs. (32-33))

$$u = \sqrt{M_2}\ \nu \tag{57}$$

$$\tau = \sqrt{M_2}^{\,3}\ \theta \tag{58}$$

with $M_2 = M_e$ for electrons and $M_2 = M_{ion}$ for the ions, but for the ions, we use eqs. (57) and (58) scaled accordingly. Combining the two sets of equations, we obtain first for the velocity squared

$$\frac{\partial}{\partial\theta} \langle\nu^2\rangle = - \frac{16}{3\sqrt{\pi}} M_1 \sqrt{M_e} \langle\nu^2\rangle + \frac{8}{\sqrt{\pi}} \sqrt{M_e} - 4 \frac{M_1}{M_{ion}} \langle\frac{1}{\nu}\rangle \tag{59}$$

In order to find an approximate solution, we neglect the constant (thermal) term $\frac{8}{\sqrt{\pi}} \sqrt{M_e}$ and substitute $\langle\nu^2\rangle^{-1/2}$ for $\langle 1/\nu\rangle$. The result integrates to

$$\langle\nu^2\rangle \simeq \left[\left\{\nu_o^{3/2} + (3\sqrt{\pi}/4)(1/(M_{ion}\sqrt{M_e}))\right\} e^{-8M_1\sqrt{M_e}/\sqrt{\pi}\cdot\theta} - (3\sqrt{\pi}/4)(1/M_{ion}\sqrt{M_e}))\right]^{2/3} \tag{60}$$

which is, of course, valid only as long as the content of the bracket is positive and far from 0. In order to evaluate orders of magnitude, we only have to remember that

$$0.2 \lesssim \sqrt{M_e}\ \nu_o \lesssim 1 \tag{61}$$

(as was noted in Figure 1). Thus the constant term is indeed a small perturbation on the solution that obtains when both of the small terms in eq. (59) are neglected.

The present discussion permits an estimate of the relative energy deposited into electrons and into ions. At any time, the fraction f of the energy

$$f = \frac{4\frac{M_1}{M_{ion}} \langle\frac{1}{\nu}\rangle}{4\frac{M_1}{M_{ion}} \langle\frac{1}{\nu}\rangle + \frac{16}{3\sqrt{\pi}} M_1\sqrt{M_e}\langle\nu^2\rangle} \tag{62}$$

goes into the ions. Moreover, the energy loss per unit θ is $\frac{\partial}{\partial\theta}\langle\nu^2\rangle$, and again we use the approximation $1/\nu\rangle \simeq 1/\sqrt{\langle\nu^2\rangle}$ *. We then find that the energy going into the ions is measured by

$$\begin{aligned}
&\int f\frac{\partial}{\partial\theta} \langle\nu^2\rangle_{ions}\, d\theta \\
&\cong \int_o^{\nu_o^2} \frac{d\langle\nu^2\rangle}{1 + \frac{4}{3\sqrt{\pi}} M_{ion}\sqrt{M_e}\langle\nu^2\rangle^{3/2}} \\
&= \frac{1}{\left[\frac{4}{3\sqrt{\pi}} M_{ion}\sqrt{M_e}\right]^{2/3}} \int_o^{\left[\frac{4}{3\sqrt{\pi}} M_{ion} \sqrt{M_e}\right]^{2/3}\nu_o} \frac{dx^2}{1 + x^3}
\end{aligned} \tag{63}$$

The integral here is easily evaluated. We have

$$\begin{aligned}
\int_o^u \frac{dx^2}{1+x^3} &= 2\left[\frac{1}{6} \ln\left[(u^2-u+1)/(u+1)^2\right]\right. \\
&\quad \left. + \frac{2}{\sqrt{3}}\left[\frac{\pi}{6} + \tan^{-1}\frac{2u-1}{\sqrt{3}}\right]\right] \\
&\rightarrow \frac{4\pi}{3\sqrt{3}} \simeq 2.4
\end{aligned} \tag{64}$$

for (as here) large values of the upper limit. This means that the energy deposited into the energy of the ions is $2.4\ E_{crit}$.

---

*We are mainly interested in ν^2 values which are an order of magnitude above thermal energies where $\langle\frac{1}{\nu}\rangle\sqrt{\langle\nu^2\rangle} = \sqrt{6/\pi} \simeq 1.38$, i.e. we only neglect a few per cent.

DISPERSION IN A TWO COMPONENT PLASMA

We now write down the equation for $\frac{\partial}{\partial\theta}\langle\nu^4\rangle$ corresponding to eq. (59) for $\frac{\partial}{\partial\theta}\langle\nu^2\rangle$, both for the interaction with electrons and with ions. The result is

$$\frac{\partial}{\partial\theta}\langle\nu^4\rangle = -\frac{32}{3\sqrt{\pi}}\, M_1\sqrt{M_e}\left[\langle\nu^4\rangle - \frac{5}{2M_1}\langle\nu^2\rangle\right]$$

$$-\, 8\,\frac{M_1}{M_{ion}}\langle\nu\rangle + \frac{8}{M_{ion}}\langle\frac{1}{\nu}\rangle \quad . \tag{65}$$

Combining (59) and (64) we obtain

$$\frac{\partial}{\partial\theta}\left[\langle\nu^4\rangle - \langle\nu^2\rangle^2\right] =$$

$$-\,\frac{32}{3\sqrt{\pi}}\, M_1\sqrt{M_e}\left\{\left[\langle\nu^4\rangle - \langle\nu^2\rangle^2\right] + \frac{1}{M_1}\langle\nu^2\rangle\right\}$$

$$-\, 8\frac{M_1}{M_{ion}}\left[\langle\nu\rangle - \langle\frac{1}{\nu}\rangle\ \langle\nu^2\rangle\right] + \frac{8}{M_{ion}}\langle\frac{1}{\nu}\rangle \tag{66}$$

which we shall separate into decay terms and production or source terms as follows (again using our standard rough approximation inserting $1/\nu\rangle \approx \langle\nu^2\rangle/\langle\nu^2\rangle^{3/2}$ and $\langle\nu\rangle\approx\langle\nu^4\rangle/\langle\nu^2\rangle^{3/2}$

$$\frac{\partial}{\partial\,\theta}\left[\langle\nu^4\rangle - \langle\nu^2\rangle^2\right] \cong$$

$$-\left[\frac{32}{3\sqrt{\pi}}\, M_1\sqrt{M_e} + 8\,\frac{M_1}{M_{ion}}\,\frac{1}{\langle\nu^2\rangle^{3/2}}\right]\left[\langle\nu^4\rangle - \langle\nu^2\rangle^2\right]$$

$$+\left[\frac{32}{3\sqrt{\pi}}\,\sqrt{M_e} + 8\,\frac{1}{M_{ion}}\,\frac{1}{\langle\nu^2\rangle^{3/2}}\right]\langle\nu^2\rangle. \tag{67}$$

It is now very interesting to compare this expression with eq. (59) neglecting the constant term (yielding the final thermal equilibrium value). The striking fact is that the decay "constant" varies with time and so does the production term in eq. (67). However, these three rates, i.e. the decay rate of $\langle v^2 \rangle$, the decay rate of the energy spread, and the production "constant" for energy spread $\simeq \langle v^2 \rangle$ are strictly proportional at any time! Thus the slowing-down process takes place exponentially, as given in eq. (60), until E_{crit} is reached, whereupon the rest of the energy is relatively quickly dumped into the ions and thermalization is attained. At the same time dispersion builds up to the small equilibrium value known from α-particle straggling, and it decays as mean energy decays. At the time when the remaining energy is dumped abruptly into ion-energy, dispersion follows suit with proportional rates for both source-terms and decay terms*. Matters change on nearing thermalization, but not drastically when one simply adds the final equilibrium values to the solutions of the homogeneous equations.

If we stick to thermal equilibrium, there is not much more to say. We shall examine the points that have been neglected: the variation of log Λ, variation of density, temperature, etc., and the neglect of small terms - also noting that we may never reach thermal equilibrium and thus we should discuss non-Maxwellian densities.

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* In other words: by an integral transform to a new time-like variable, our equations become those of simple exponential decays.

NON-MAXWELLIAN DENSITIES

So far we have relied heavily on properties of the Maxwellian distribution of our field particles, and thus we were able to establish the relations eqs. (35-42), which were essential in the reduction of the Fokker-Planck equation to the form of eq. (43). Behind these equations also lies the fact that the Coulomb logarithm has been considered independent of relative velocity. This latter point is, however, of no great importance, and a special numerical investigation of it is described in Appendix B. Here we shall discuss non-Maxwellian field particles - but retain the $\log \Lambda$ = const idea. In this case we have quite generally

$$J_2(u) = -\frac{1}{2}\frac{\partial}{\partial u} J_1(u). \tag{68}$$

The reason behind this is the fact that

$$w\partial w = (v_1 - v_2 \cos\theta)\partial v_1 , \tag{69}$$

and that precisely w^2 and $(v_1-v_2\cos\theta)$ occur fundamentally in our friction and diffusion coefficients (cf. eq. A 15), and that our scattering law is $\sim 1/w^3$.

We now use the definitions (36, 37) of G and H, and construct from equation (34) the equation for the second moment. We find (without reference to any Maxwellian distribution for the field particle) with J_1 a generalized integral (eq. A 1)

$$\frac{\partial}{\partial\tau}\langle u^2\rangle = 2\int \frac{M_1+M_2}{M_2} u^3 F \frac{\partial J_1}{\partial u} du + 2 \int u^2 F J_1 du \tag{70}$$

we then use this for electron cooling and note that J_1 has the form

$$J_1 = A(1 - au^2 + ..) \equiv \langle\frac{1}{u^2}\rangle - \frac{1}{6} P_o u^2 + .. \tag{71}$$

Remembering that $M_2/M_1 <<1$, we find

$$\frac{\partial}{\partial\tau}\langle u^2\rangle \simeq - 4\frac{M_1}{M_2} aA\langle u^2\rangle + 2A \tag{72}$$

with the solution

$$\langle u^2 \rangle \simeq \frac{1}{2}\,\frac{M_2}{M_1}\,\frac{1}{a} + u_o{}^2\, e^{-4\frac{M_1}{M_2}\, aA\tau}$$

i.e. the "decay" rate of the energy is given by

$$4\,\frac{M_1}{M_2}\;aA = \frac{2}{3}\,\frac{M_1}{M_2}\,P_o \tag{74}$$

where P_o is the u_2=o value of the u_2 distribution, and where our approximations assume a certain measure of wellbehavedness of the field particles. Granted all that, we find (not very surprisingly) that the cooling rate is directly proportional to the number of cool field particles, $P_o = P(u^2{=}o)$. Stated differently we may say that for a given average energy of the field particles we gain in cooling-power by having a velocity distribution for the field particles with more zero velocity particles (and then necessarily a longer high energy tail) than the Maxwellian. This is, of cource, a direct consequence of the fact that the friction function J_2 only results from particles with $u_2 \leq u_1$ (see eq. (A13) of Appendix A). The argument raised here breaks down when so many particles are shifted away from average (partly to low energy and partly to the high energy tail) that eq. (71) becomes a poor approximation.

GENERAL CONSIDERATIONS ON THE NON-MAXWELLIAN DISTRIBUTION

A few interesting points can be shown in the case when the distributions P are arbitrary. In particular, it is possible to discuss relatively simply the non-linear case when an initial δ-function-like velocity distribution of particles thermalizes on itself through Coulomb interaction[6]. The non-linearity in this case results from putting P = F. In this case the Fokker-Planck equation becomes an integro-differential equation. First, however, we deal with the case of a test particle interacting with a non-Maxwellian ensemble of field particles. Corresponding to eq. (43), we find

$$\frac{\partial F}{\partial \tau} = \frac{1}{u^2}\frac{\partial}{\partial u}\, u^2\left(J_3\,\frac{\partial}{\partial u}\,F + \frac{M_1}{M_2}\,2J_2\,F\right) \tag{75}$$

which yields eq. (43) in the special case when $J_2 = u\,J_3$ (see eq. (A26)), which is a particular property of Coulomb interaction with a Maxwellian plasma.

In order to prove eq. (75), one has to use eqs. (A11-14) and the generalized definitions of the functions J_i. One can then prove that

$$2u^2J_2F + u^2F\frac{\partial}{\partial u}J_3 - 2uFH = 0. \tag{76}$$

The particularly interesting case of self-thermalization results when P = F in the definitions of J_i and when $M_1 = M_2$.

We find

$$\frac{\partial F}{\partial \tau} = \frac{1}{u^2}\frac{\partial}{\partial u}\left\{u^2\left[\frac{\partial F}{\partial u}\left(\frac{2}{3u^3}\int_o^u F(v)\,v^4dv \right.\right.\right.$$
$$\left.\left.\left. + \frac{2}{3}\int_u^\infty F(v)\,v\,dv\right) + 2F\,\frac{1}{u^2}\int_o^u F(v)\,v^2dv\right]\right\} \tag{77}$$

By partial integration this equation permits us to prove the conservation of probability and energy, i.e.

$$\frac{\partial}{\partial\tau}\left(\int_0^\infty u^2F\,du\right) = 0 \Rightarrow \int_0^\infty u^2F du = 1 \tag{78}$$

$$\frac{\partial}{\partial\tau}\left(\int_0^\infty u^4F du\right) = 0 \Rightarrow \int_0^\infty u^4F du = \frac{3}{2} \tag{79}$$

where the numerical constants 1 and 3/2 are chosen for convenience.

From equation (75) we may finally extract the result that $F \simeq e^{-u^2}$ is a selfconsistent solution for the equilibrium distribution $\partial F/\partial t = 0$. Since, according to Lindhard and Nielsen[7)], the equilibrium distribution is not only a solution but the unique equilibrium solution, we see the significance of this result.

Now we must remember that we made a few approximations. Notably we have used binary collision theory along the average (eikonal) orbit of the test particle and neglected orbit fluctuations. Also we truncated the Coulomb logarithm by a velocity-independent argument, i.e., velocity-dependent Debye length λ, in exactly such a way as to cancel the velocity dependence of the closest encounter, and with this constraint imposed we used $\lambda_D \to \infty$ in order to neglect "small terms".

The author finds it interesting that this simultaneous set of approximations yields such physically relevant equations concerning the thermalization process.

CONCLUSIONS

The conclusions of the present investigation are that a survey of the theory of cooling of α-particles in a thermonuclear plasma can be understood to quite a detailed extent by analytical computations pertaining to an isotropic, homogeneous plasma.

The α-particle energy will primarily go into heating of the electron component of the plasma and only some per cent of the energy goes into the ion component; even so this only happens late in the slowing-down process - so late that there may arise serious competition with confinement times.

Near-thermal ions exchange energy at a fast rate among themselves, and the rate is even faster for near-thermal electrons among themselves. However, the energy-exchange rate between ions and electrons is usually* slow. Thus, sadly enough, the α-particle energy will mainly go into electron energy at a slow rate (then abruptly at a fast rate some per cent directly to the ions), and then from there into an electron heating of the now less heated ion-component, but again at a slow rate. The energy transfer is well represented by exponentials in time. The rate constants have been derived in the above.

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* For temperatures, densities, etc., of interest in thermonuclear plasmas.

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Appendix A

The three integrals

$$J_1 = \int_0^\infty P(v_2)\ v_2^2\ dv_2 \int_{-1}^{1} \frac{1}{2}\ d\cos\phi\ 2w^2/w^3 \tag{A1}$$

$$J_2 = \int_0^\infty P(v_2)\ v_2^2\ dv_2 \int_{-1}^{1} \frac{1}{2}\ d\cos\phi$$

$$(v_1 - v_2 \cos\phi)/w^3 \tag{A2}$$

$$J_3 = \int_0^\infty P(v_2)\ v_2^2\ dv_2 \int_{-1}^{1} \frac{1}{2}\ d\cos\phi$$

$$v_2^2 \sin^2\phi/w^3 \tag{A3}$$

where

$$w^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\phi \tag{A4}$$

may be transformed into integrals over dw with

$$w_- \leq w \leq w_+ \tag{A5}$$

where

$$w_- = |v_1 - v_2| \tag{A6}$$

$$v_+ = v_1 + v_2 \tag{A7}$$

using

$$v_2^2 \sin^2\phi = [w^2 - w_-^2]\ [w_+^2 - w^2]/(4v_1^2) \tag{A8}$$

$$\partial \cos\phi / \phi w = - w/(v_1 \; v_2) \tag{A9}$$

$$v_1 - v_2 \cos\phi = (w^2 + v_1^{\;2} - v_2^{\;2})/2\, v_1 \tag{A10}$$

The results of the integrations over w ($\cos\phi$) become

$$J_i = \int_o^\infty P(v_2)\; v_2^{\;2}\; dv_2\; j_i\; (v_1, v_2) \tag{A11}$$

with

$$j_1 = \begin{cases} 2/v_1 & o < v_2 < v_1 \\ 2/v_2 & v_1 < v_2 < \infty \end{cases} \tag{A12}$$

$$j_2 = \begin{cases} 2/(2v_1^{\;2}) & o < v_2 < v_1 \\ o & v_1 < v_2 < \infty \end{cases} \tag{A13}$$

$$j_3 = \begin{cases} \frac{2}{3}\; v_2^{\;2}/v_1^{\;3} & o < v_2 < v_1 \\ \frac{2}{3}\; 1/v_2 & v_1 < v_2 < \infty \end{cases} \tag{A14}$$

Equation (A13) indicates that only particles slower than the test particle (alfa-particle in our case) contribute to the friction. From (A12-13), or directly from (A1-2), one notes

$$j_2 = - \frac{1}{2} \frac{\partial}{\partial v_1}\; j_1 \; . \tag{A15}$$

For the special case where $P(v_2)$ is a Maxwellian distribution, we find the following results

$$J_i = \frac{4}{\sqrt{\pi}} \int_o^\infty e^{-v_2^{\;2}}\; v_2^{\;2}\; dv_2\; j_i \tag{A16}$$

$$J_1 = \frac{2}{v_1} \text{ERF } (v_1) \tag{A17}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{m=o}^{\infty} \frac{(-1)^m}{m!} \frac{1}{2m+1} v_1^{2m} \tag{A18}$$

$$\rightarrow 2/v_1 \qquad \text{for} \quad v_1 \rightarrow \infty \tag{A19}$$

$$J_2 = \frac{1}{v_1^2} \text{ERF } (v_1) - \frac{2}{\sqrt{\pi}} \frac{1}{v_1} e^{-v_1^2} \tag{A20}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{m=o}^{\infty} \frac{(-1)^m}{m!} \frac{1}{2m+3} v_1^{2m+1} \tag{A21}$$

$$\rightarrow 1/v_1^2 \qquad \text{for} \quad v_1 \rightarrow \infty \tag{A22}$$

$$J_3 = \frac{1}{v_1^3} \text{ERF } (v_1) - \frac{2}{\sqrt{\pi}} \frac{1}{v_1^2} e^{-v_1^2} \tag{A23}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{m=o}^{\infty} \frac{(-1)^m}{m!} \frac{1}{2m+3} v_1^{2m} \tag{A24}$$

$$\rightarrow 1/v_1^3 \qquad \text{for} \quad v_1 \rightarrow \infty \tag{A25}$$

Note that in this case

$$J_3 = \frac{1}{v_1} J_2 = - \frac{1}{2v_1} \frac{\partial}{\partial v_1} J_1 \tag{26}$$

and equation (42) of the text can be proved inserting eqs. (A17, A20, A23) into

$$\frac{\partial}{\partial v_1} J_3 - \frac{1}{v_1} (J_1 - 3J_3) + 2v_1 J_3 = 0$$

and proving the equality.

Appendix B

In the main text we have considered log Λ to be a constant. However, the forms of eq. (29) are

$$\log \Lambda = \log(1 + Aw^4) \tag{B1}$$

and

$$\log \Lambda = \log(1 + Bw^2). \tag{B2}$$

Consequently, the integrals (A1-A3) with such factors included may, through eqs. (A8-10), be considered as consisting of sums of integrals of the form

$$J = \int_{w_-}^{w_+} w^m \log(1 + A\, w^n)\, dw \tag{B3}$$

which again is a sum of n integrals of the form (scaled indefinite integrals)

$$J_j = \frac{1}{\alpha_j^{m+1}} \int (\alpha_j w)^m \log(1 + \alpha_j w)\, d(\alpha_j w) \tag{B4}$$

with α_j the j^{th} root of $\sqrt[n]{-A}$.

However, these integrals are easily calculated through repeated integrations by parts. The final step is then a numerical integration over $P(v_2)$. Thus everything can be done, and the following numerical investigation was carried out.

The complete integrals corresponding to our three coefficients of friction and diffusion were calculated and divided by

$$\log\Lambda_{const} = \log(4\lambda_D{}^2/(Z_1Z_2M_2/\mu\cdot(511/T))^2\cdot 2.8\cdot 10^{-13} \tag{B5}$$

with λ_D in cm and T in keV. (λ_D from eq. (27)).

The coefficients were then compared to a standard set computed from eqs. (A17, A20, A23) and the deviation is found as

$$\text{Deviation} = 100 \frac{J - J_{\Lambda=\text{const}}}{J_{\Lambda=\text{const}}} \tag{B6}$$

and is consequently given in per cent of the Λ = const values. Several parameters were varied: λ_D could be multiplied by a factor α, the transition of Λ from a classical to a quantum-mechanical formula could be varied by a factor β multiplied into the left side of eq. (31). A factor γ was introduced in front of the small terms (eq. (26)), and $\gamma=1$ means small terms included and $\gamma=0$ small terms neglected. Small terms were calculated from the formulas of appendix A since the Coulomb integral for them was equated to 2 as mentioned in connection with eq. (26). Also temperature, density and mass M_2 of the field particles was varied. Finally P_2 could be changed.

The first results to be presented are a standard set of functions J_2, J_3, G and H derived from equations (A17-25) and (36, 37), i.e. the standard scaled functions. Remember that $J_3=J_2/u$. The results are shown in Fig. B1. In all our drawings u is varied by a factor of 4 per unit in the figures, thus the functions are depicted in log scale for a variation of u by a factor $\sim 10^3$, (in Fig. B1, u is varied by $5 \cdot 10^3$).

Clearly, the asymptotic behaviour is changed when $\Lambda \sim \log w$ is included, roughly speaking, into

$$J \simeq \log u/u^n \tag{B7}$$

Thus all the curves will show larger deviations for large values of v.

A reference case of T = 1 keV, $M_1 = 4$, $M_2 = 1$, $Z_1 = 2$, $Z_2 = 1$, $\alpha = \beta = j = 1$ and $n = 10^{14}$ is shown in Fig. B2. Here we see a typical result, deviations of minus a few per cent for $u \lesssim 1$ and the log u dependence for $u \gtrsim 1$. The main coefficient is J_3, which goes as $\log u/u^3$ rather than $1/u^3$. Consequently, when J_3 has practically disappeared, it is changed by some several per cent all told.

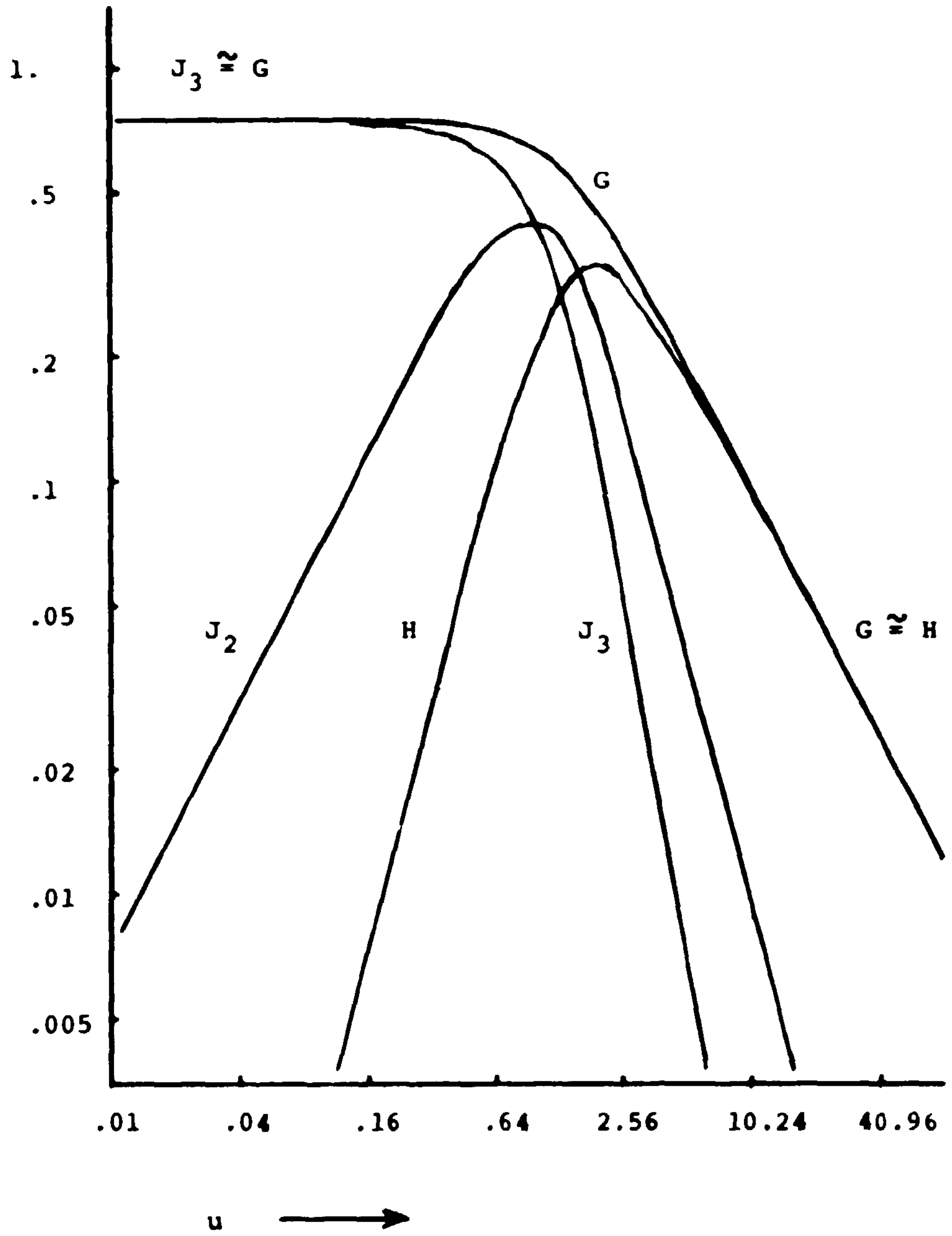


Fig. B1. The standard set of functions J_2, J_3, G, and H obtained from eqs. (A17-25) and the definitions eqs. (36, 37) of G and H.

The varied cases are shown in Figs. B3-B8.

Finally, a non-Maxwellian distribution

$$P(u_2) = 4\sqrt{8}\, \exp(-\sqrt{8}\, u_2)$$

was introduced with the same normalization and average energy as our standard Maxwellian, but with P(o) roughly 5 times as large as for the Maxwellian. Consequently, H and J_2 deviate by a similar factor for $u < 1$, i.e. for the region in u which is interesting for electron cooling. Figure B9 illustrates the $u > 1$ behaviour.

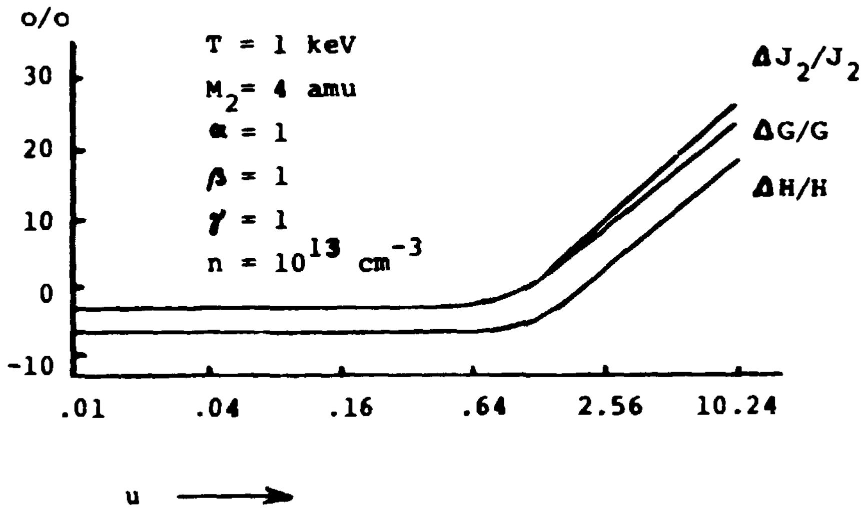
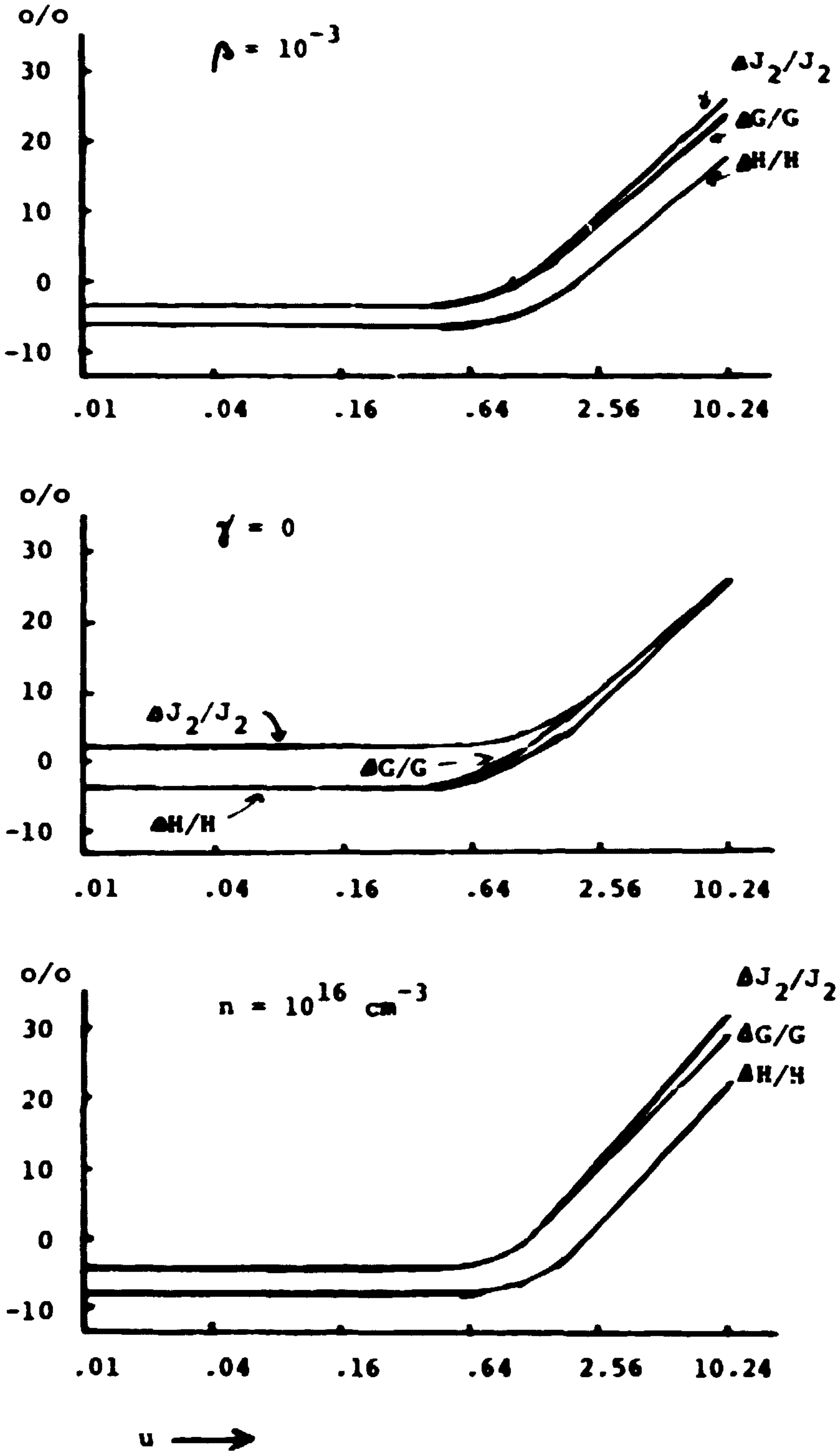
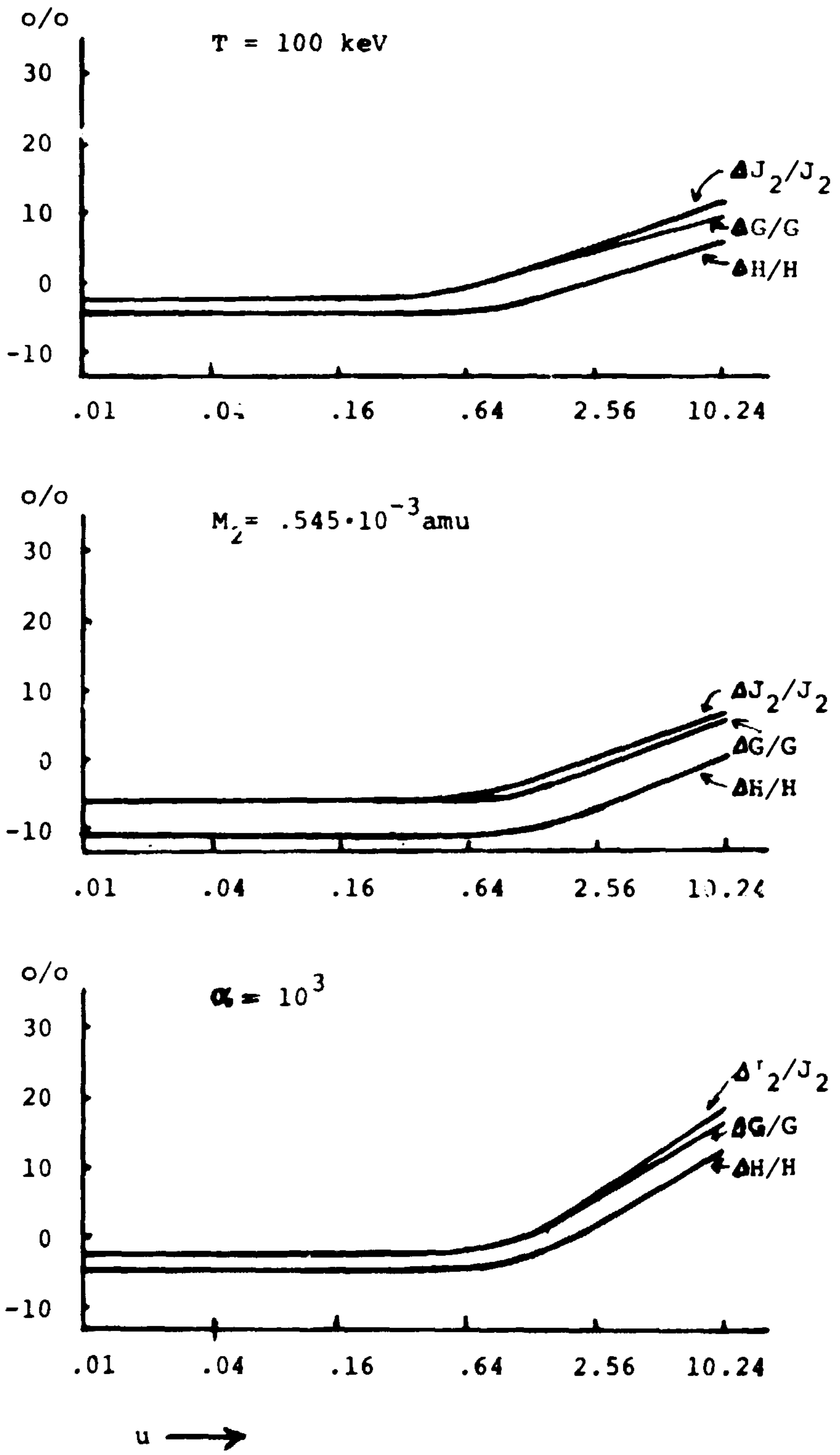


Fig. B2. The values of $\Delta J_2/J_2$, $\Delta G/G$ and $\Delta H/H$ in % and as functions of u for the reference case of input data as noted in the figure. (Also M_1 = 4 amu, Z_1 = 2 appropriate for alfa-particles.



Figs. B3-5. Similar to Fig. B2 but with the individual parameters changed as indicated.



Figs. B6-8. Similar to Fig. B2 but with the individual parameters changed as indicated.

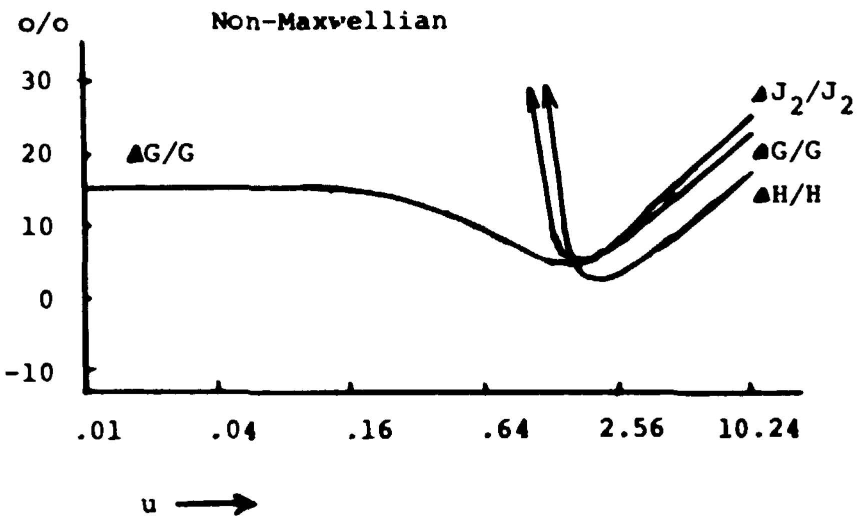


Fig. B9. The non-Maxwellian results as mentioned in the text.

Appendix C

The text contains statements to the effect that electron cooling is well described by the longest "timeconstant" which to a good approximation is given by

$$1/\bar{\tau} = \frac{16}{3\sqrt{\pi}} \frac{M_1}{M_2} \tag{C1}$$

and also that the remaining energy is eventually dumped "relatively quickly" into the ion component of the plasma, i.e. relatively quickly compared to $\bar{\tau}$. These statements are based on the following study of thermalization in the $M_2/M_1 << 1$ approximation. Our starting point is the Fokker Planck equation as given by eq. (43) with J_3 expanded in u^2 as shown by eq. 47 (eq. (A24) of appendix A). We now proceed step by step and start with

$$J_3 = \frac{4}{3\sqrt{\pi}} \tag{C2}$$

and we seek solutions to the Fokker Planck equation which are separable in τ and u, i.e. we put

$$F = \psi_n(u) \cdot \theta_n(\tau) \tag{C3}$$

with n a possible eigenvalue parameter belonging to our problem. This leads to the separated equations

$$\frac{d}{d\tau} \theta = K_n \theta \tag{C4}$$

and

$$\frac{4}{3\sqrt{\pi}} \frac{1}{u^2} \frac{d}{du} \left[u^2\left(\frac{d\psi_n}{du} + 2u \frac{M_1}{M_2} \psi_n\right)\right] = K_n \psi_n. \tag{C5}$$

For convenience, we introduce

$$v = u\sqrt{\frac{M_1}{M_2}} \tag{C6}$$

$$C_n = \frac{3\sqrt{\pi}}{4} \frac{M_2}{M_1} K_n \tag{C7}$$

and

$$\psi_n = \phi_n e^{-v^2} \tag{C8}$$

leading to

$$\frac{d^2}{dv^2}\phi_n + 2\left[\frac{1}{v} - v\right]\frac{d\phi_n}{dv} - C_n\phi_n = 0. \tag{C9}$$

This equation has the following set of eigenfunction solutions (which are wellbehaved for $v \to 0$) for the eigenvalues $C_n = -4n$, $n = 0, 1, 2, \ldots$

$$\phi_n = 1 + \sum_{m=1}^{n} a_m v^{2m} \tag{C10}$$

with (for each value of n)

$$a_{m+1}(2m+2)(2m+3) = a_m(C_n + 4m) \tag{C11}$$

The first few polynomials and eigenvalues are

$$\begin{aligned} \phi_o &= 1 && \text{for } C_o = 0 \\ \phi_1 &= 1 - \tfrac{2}{3}v^2 && \text{for } C_1 = -4 \\ \phi_2 &= 1 - \tfrac{4}{3}v^2 + \tfrac{4}{15}v^4 && \text{for } C_2 = -8 \end{aligned} \tag{C12}$$

etc.

From this it follows that we can decompose any initial distribution F(o,u) in such a way that

$$F(\tau,u) = \sum_{n=o} a_n e^{-\frac{16n\ M_1}{3\sqrt{\pi}\ M_2}\tau} \cdot \phi_n\left(u\sqrt{\frac{M_1}{M_2}}\right) \cdot e^{-\frac{M_1}{M_2}u^2} \tag{C13}$$

We shall illustrate this point later on, but first consider the next approximation where small variations of J_3 with u are included. We write

$$J_3 = \frac{4}{3\sqrt{\pi}} \left(1 - \frac{3}{5}\frac{M_2}{M_1}v^2 + \frac{3}{14}\left(\frac{M_2}{M_1}\right)^2 v^4 - ..\right)$$

$$\equiv \frac{4}{3\sqrt{\pi}} J$$

$$\equiv \frac{4}{3\sqrt{\pi}}(1 - \beta_1 v^2 + \beta_2 v^4 - \ldots) \tag{C14}$$

Corresponding to eq. (C9) we find in this case

$$J\frac{d^2}{dv^2}\phi_n + \left[2J(\frac{1}{v} - v) + \frac{dJ}{dv}\right]\frac{d\phi_n}{dv} - C_n\phi_n = 0 \tag{C15}$$

which is not a simple equation like eq. (C9). Here we shall only solve approximatively considering $\beta_2 = 0$ and β_1 a small perturbation. We shall find that we are admixing neighbouring eigenfunctions of eq.(C9),but in small amounts only proportional to β_1. Thus, if we insert solutions of the form eq. (C10), then we may, e.g., try to find a solution close to ϕ_1 of eq. (C12) by inserting $a_2 \simeq a\beta_1$ and neglecting higher order terms like β_1^2, a_3, a_4, etc. The result is

$$C_1 \simeq -4 + 6\frac{M_2}{M_1}$$
$$a_1 \simeq -\frac{2}{3} + \frac{M_2}{M_1} \tag{C16}$$

and

$$a_2 \simeq -\frac{2}{5}\frac{M_2}{M_1}$$

which justify the $\frac{M_2}{M_1} \to 0$ approximation for electron cooling.

We now return to the $\beta_1 = \beta_2 = 0$ approximation. In this particular case, with the solutions to the Fokker Planck equation given by eq. (C13), one may show that the following expression

$$F(T,v) = \sum_{m=o}^{N} B_{N,m}(-1)^m\phi_m(v)e^{-v^2}\ e^{-mT} \tag{C17}$$

written in the v, $T = \tau 16 M_1/(3\sqrt{\pi} M_2)$ system and with $B_{N,m}$ the binomial coefficients, is a solution which satisfies the initial condition

$$F(o,v) = \frac{2^N}{(2N+1)!!} v^{2N} e^{-v^2} \tag{C18}$$

which for large N approximates a δ-function around $v = \sqrt{N}$. Thus eq. (C17) is an analytical approximation (and for $M_2/M_1 << 1$ a very good approximation for velocities much smaller than the thermal velocity of the field particles) to the entire slowing-down and thermalization problem. Figure C1 illustrates the N= 10 case as a function of T and v.

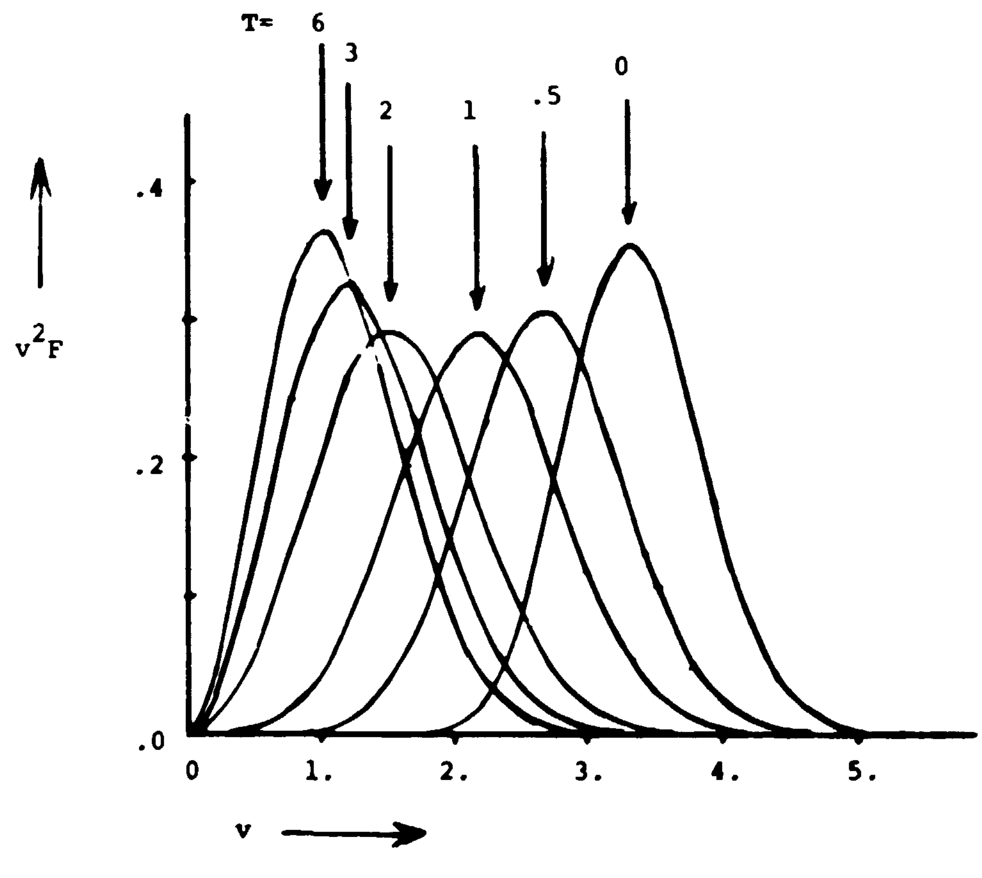


Fig. C1. The velocity distribution v^2F as a function of T and v and as given by eq. (C17) and for N = 10.

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